Homological stability

Type: PhD course, basic
Sector: MAT/02, MAT/03
Lecturer: L. Caputi
Number of lectures: 16 hours, for a total of 8 meetings
Period: January – March 2026
Exams: seminar form
Preliminaries: Notions of homology groups and basic notions of category theory

Description. The course will introduce the students to the basic theory and results in both homological stability and representation stability. The course will also survey recent developments of the theory and current research directions in the field.

Abstract. Homology is an algebraic invariant that can be defined for several objects: groups, (topological) spaces, algebras, categories. We will be mainly interested in homological invariants of *families* of groups, and in their "stable" homology groups. In fact, homological stability has shown that, for certain families, it is easier to compute infinitely many homology groups at once rather than computing single homology groups proceeding step by step one after the other. Homological stability can be described as follows. Suppose that

$$G_0 \to G_1 \to \ldots \to G_N \to \ldots$$

is a sequence of groups and homomorphisms between them. We say that the family $\{G_n\}$ satisfies homological stability if for each q the induced maps $H_q(G_n) \to H_q(G_{n+1})$ are isomorphisms for all $q \leq f(n)$ where $f(n) \to \infty$ with n increasing. Analogously, we can ask the same property for families of topological spaces, algebras, etc. Then, once homological stability for a family is satisfied, one can compute homology groups in the stable range at once. Furthermore, the homology of the limit group can be tackled with homotopy theoretic methods, tools which are inaccessible in the unstable range.

We will deal with families of groups from both algebra and topology, such as the symmetric groups, the braid groups, and the mapping class groups. In fact, homological stability was shown to be satisfied for various families: symmetric groups, braid groups, mapping class groups of surfaces, general linear groups, orthogonal groups, diffeomorphism groups of highly-connected high-dimensional manifolds, to name a few of a big list. Stability techniques in homology of groups started with the intriguing work by Quillen [Qui72] on the computation of K-theory of finite fields; which was also the main motivation. His proof, nowadays known as the *Quillen argument*, ignited the interest in stability phenomena also in other fields, with many new questions arising in geometric topology.

The main aim of the course is to introduce the fundamental concepts of homological stability, the Quillen argument, to provide the proofs of the main cases, and then to survey similar ideas

in related fields; for example, in representation theory. If time allows, we will conclude the course with showing some recent applications of the theory in the homology of matching complexes and configuration spaces of graphs.

Tentative program.

- Lecture 1 & 2: In these first lectures we first introduce the general concept of homological stability. We will introduce some basic concepts and definitions, such as of group homology and of simplicial complexes, needed in the course.
- Lecture 3 & 4: we start, as warm-up, with the proof of homological stability of braid groups. In particular, we will study the conectivity of the curve complex.
- Lecture 5 & 6: We outline Quillen's argument, which is a spectral sequences argument. Therefore, we will first recall some basic notions of spectral sequences needed to carry out the argument, and then provide the proof. If time allows, we will also describe the stable homology of braid groups.
- Lecture 7 & 8, 9& 10: Homological stability can be generalized to various automorphism groups of objects living in (well-behaved) categories. In these lectures, we will generalize the constructions for braid groups to more general automorphism groups. We will use the categorical language of homogeneous categories, which will be introduced and investigated with examples; such as general linear groups and mapping class groups. We will follow the categorical framework due to Randal-Williams and Wahl [RWW17].
- Lecture 11 & 12. In this lecture, we will be interested in representation stability problems [Far14, CF13]. We will describe the difference between homological stability and representation stability, the motivation, and discuss the main examples of configuration spaces and pure braid groups. Our driving examples will be groups of configurations of points in the disc. If time allows, we will discuss the case of Torelli groups.
- Lecture 13 & 14. Representation stability is related to the question whether a certain category of objects is Noetherian; ie, that subfunctors of finitely generated functors are finitely generated. We will intrduce the concept of FI-modules, and investigate the connections with representation stability.
- Lecture 15 & 16. A recent theory developed by Sam and Snowden [SS17] tackles the problem of noetherianity of functors from a combinatorial point of view. We will introduce the notion of Gröbner categories, and discuss the relation between Noetherianity of categories and Gröbner categories. We will apply the theory developed by Sam and Snowden to the category of graphs, and we will see that, when bounding the genus, (representation categories of) categories of graphs are Noetherian. We will apply it to the case of matching complexes and magnitude homology.

If time allows, in these lectures, we will also survey more recent developments in homological stability, a topic that is still growing and attracting interest.

References

[CF13] Thomas Church and Benson Farb. Representation theory and homological stability. *Advances in Mathematics*, 245:250–314, 2013.

[Far14] Benson Farb. Representation stability. *arXiv preprint arXiv:1404.4065*, 2014.

[Qui72] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. Math. (2)*, 96:552–586, 1972.

[RWW17] Oscar Randal-Williams and Nathalie Wahl. Homological stability for automorphism groups. *Adv. Math.*, 318:534–626, 2017.

[SS17] S. V. Sam and A. Snowden. Gröbner methods for representations of combinatorial categories. *J. Amer. Math. Soc.*, 30(1):159 – 203, 2017.